UTS-DFT-94-09

Postmodern String Theory: Stochastic Formulation

A.Aurilia

Department of Physics, California State Polytechnic University, Pomona, CA 91768

E.Spallucci

Dipartimento di Fisica Teorica Università di Trieste, INFN Sezione di Trieste, Strada Costiera 11, 34014 Trieste, Italy

I.Vanzetta

Dipartimento di Fisica Teorica Università di Trieste, Strada Costiera 11, 34014 Trieste, Italy

Abstract

In this paper we study the dynamics of a statistical ensemble of strings, building on a recently proposed gauge theory of the string geodesic field. We show that this stochastic approach is equivalent to the Carathéodory formulation of the Nambu-Goto action, supplemented by an averaging procedure over the family of classical string world-sheets which are solutions of the equation of motion. In this new framework, the string geodesic field is reinterpreted as the Gibbs current density associated with the string statistical ensemble. Next, we show that the classical field equations derived from the string gauge action, can be obtained as the semi-classical limit of the string functional wave equation. For closed strings, the wave equation itself is completely analogous to the Wheeler-DeWitt equation used in quantum cosmology. Thus, in the string case, the wave function has support on the space of all possible spatial loop configurations. Finally, we show that the string distribution induces a multi-phase, or *cellular* structure on the spacetime manifold characterized by domains with a purely Riemannian geometry separated by domain walls over which there exists a predominantly Weyl geometry.

PACS number: 11.17

I. INTRODUCTION AND SYNOPSIS

The term *postmodern*, frequently encountered in art history, has been lately applied to quantum mechanics to describe the renewed interest in early semiclassical approximation schemes. In the words of Heller and Tomsovic [1], *" ... after the " modern " innovations have been assimilated, the threads of premodern thought are always reconsidered. Much of value may be rediscovered and put to new use. The modern context casts new light on premodern thought, which in turn shades perspectives on modernism."*

Those words aptly describe the spirit of this paper which has its roots in the premodern string theory of Nambu [2], and Nielsen and Olesen [3]. That theory led to the first successful description of hadrons as extended objects made up of constituent quarks. Since then QCD has established itself as the local gauge theory of quarks and gluons even though a generally accepted quantitative account of the low energy binding process of quarks into hadrons is still lacking. Over the years, the emphasis has shifted to superstrings because of their alluring, but unfulfilled promise of superunification of the fundamental forces. In more recent years, the focus of attention has been on *cosmic strings*. Here again, the tantalizing prospect is nothing short of explaining the large scale structure of the universe in the framework of the inflation-driven Big Bang theory. Even for cosmic strings there is a premodern theory, or *old picture*, based essentially on simulation codes. Admittedly, such numerical methods provide thus far the most detailed knowledge about the dynamics of a string network. However, their reliability, let alone their theoretical value, may seem somewhat questionable. Indeed, even some of the original investigators of cosmic strings have advocated the use of a more analytical approach which might corroborate the numerical results [4].

Against this background, the aim of this paper is twofold: a) to construct a new theory of relativistic strings building on the insights provided by the premodern approach but without its shortcomings, and b) to investigate the effect of the theory on the geometry of spacetime.

Central to our approach is the premise that the dynamics of a string network, cosmic or otherwise, is a random process at the outset [5]. From this statistical basis we attempt to construct a consistent theory of relativistic strings in terms of gauge fields and quantum loop variables. The long range objective of this research is perhaps best described as the formulation of a *stochastic quantum field theory of relativistic strings*, which we regard as the paradigm of a general approach to the dynamics of extended systems with any number of dimensions. The main points of the present approach can be summarized as follows :

1) Our string theory is initially defined in Minkowski space. However, we will show that, in general, the geometry of the spacetime manifold is shaped by the string distribution into a *domain*, or *cellular* structure such that domain walls with a Weyl type geometry enclose *voids*, or " pockets " of spacetime in a Riemannian phase.

2) In the quantum domain, our stochastic formulation does for strings what the Wheeler-DeWitt equation does for quantum cosmology. In the string case, loop configurations, interpreted as spatial sections of world -sheets, correspond to the spatial 3-geometries in the Wheeler-DeWitt formulation of quantum cosmology. A point-splitting regularization of the string wave equation enables us to recover the classical field equations and to relate the *quantum* probability amplitude defined in loop-space to the *classical* string distribution function. In this technical sense we speak of *stochastic* semi-classical formulation of string theory. This leads us to the next and, perhaps, most fundamental point.

3) Stochastic formulations of quantum mechanics are just as old as conventional quantum mechanics. This paper extends to relativistic strings the stochastic approach used for point particles [6]. On the technical side, the novelty of our approach stems from the use of Carathéodory's formulation of the Hamilton-Jacobi variational principle. As our analysis shows, even though the Hamilton-Jacobi principle is deeply rooted in the deterministic approach to classical mechanics, in Carathéodory's interpretation it takes on a new meaning which opens the way to the stochastic formulation suggested here. Since this point is the crux of our arguments, it seems useful to expand on it at this introductory stage. As is well known, one way to bridge the gap between the classical and quantum domain is via the correspondence between wave phenomena and geometric ray paths, and in this connection the importance of the Hamilton–Jacobi formulation of classical mechanics cannot be overemphasized. For our later purposes, it is enough to recall the following two properties, i) the Hamilton–Jacobi equations can be derived in the WKB limit from the Schrödinger equation, and, conversely, ii) the classical Hamilton–Jacobi wavefronts associated with the motion of a mechanical system represent the surfaces of constant phase of the corresponding wavefunction.

In two previous papers, upon reexamination of premodern string theory [2], [3], [7], [8], [9], [10], [11], [12], we have proposed a gauge field theory which admits extended objects as singular solutions of the field equations [13], [14]. The theory is formulated in terms of two independent field variables which, in the simplest case of a closed string, are a vector gauge potential $A_\mu(x)$, and an antisymmetric tensor $W^{\mu\nu}(x)$. Both fields acquire a well defined geometrical and physical meaning in the Hamilton–Jacobi formulation of string dynamics: once the field equations are solved, $A_\mu(x)$ acquires the meaning of vector potential associated with a family of extremal world-sheets, while $W^{\mu\nu}(x)$ plays the role of string current. The field action, once evaluated on such a solution, reproduces the Nambu-Goto action, thus connecting the gauge formulation to the usual geometrical description of the string motion.

However, from the postmodern vantage point, the fundamental action is the gauge one, while the Nambu-Goto geometric functional plays the role of *effective action* for the low-energy string dynamics. In other words, postmodern string theory is *on-shell* equivalent to the Nambu-Goto string model.

One of the objectives of this paper is to extend off shell the above on-shell equivalence between the two formulations. Furthermore, we wish to show that the classical field equations for $A_\mu(x)$ and $W^{\mu\nu}(x)$ can be derived as the WKB limit, $\hbar \to 0$, of the functional wave equation for a closed, bosonic string.

In this paper, both the off-shell equivalence between the geometric and gauge string action, and the semi-classical limit of the quantum wave equation are discussed in the framework of the Carathéodory formulation of the Hamilton–Jacobi action principle. A novel feature of our approach is an averaging procedure over the family of extremal world-sheets which are solutions of the Hamilton–Jacobi equations. This procedure introduces, at the classical level, a random element in the string evolution. Each member of a family of extremal world-sheets is labelled by a pair of parameters. We assume that the values of such parameters are not equally likely but are statistically distributed. In other words, rather than dealing with a single string, we consider a statistical ensemble of them, not unlike a relativistic fluid consisting of filamentary constituents. The probability distribution, or fluid energy density, is an assigned function at the classical level. *One of the main results of our*

analysis is to relate such a classical function to the quantum probability amplitude, hence connecting the statistical classical behavior of the system described by our gauge lagrangian, to the string quantum dynamics. In view of this result, it is tempting to interpret the string fluid as a classical model of some quantum ground state, or string condensate, leading us to a hydrodynamic picture of the string fluid evolution in which $W^{\mu\nu}$ is interpreted as a divergence-free fluid current, while A^{μ} plays the role of velocity potential.

The paper is organized as follows.

Sect. II is a pedagogical review of the Carathéodoryformulation of the Hamilton–Jacobi action principle for a relativistic point particle. It has the double purpose to introduce the relevant notation, and to " translate " some of the mathematical terminology in a language which is more familiar to physicists.

In Sect. III we discuss the Carathéodoryformulation of string classical dynamics in *Minkowski space*, and introduce the average over the family of classical extremal surfaces. The final result is that we recover the string gauge action from the Nambu-Goto area functional without the use of classical solutions.

In Sect.IV we discuss the semi-classical limit of the string functional wave equation. A point- splitting regularization is introduced which enables us to recover the classical field equations in the limit $\hbar \to 0$. The distribution function of classical extremals in parameter space is related to the square of the string wave functional, and the quantum counterpart of the classical current density is identified.

In Sect.V we show that the string distribution induces a multi-phase, or *cellular*, structure on the spacetime manifold. In this structure, domains of spacetime characterized by a Riemannian geometric phase and a nearly uniform string distribution, appear to be separated by domain walls. On these walls there exists a predominantly Weyl geometry and across them the string density changes appreciably.

Throughout the paper we shall assume metric signature $- + ++$, and express physical quantities in natural units. We use $\hbar \neq 1$ only in the semi-classical quantum formulae.

II. HAMILTON-JACOBI THEORY OF THE RELATIVISTIC POINT-PARTICLE

A. Families of extremals and wave fronts

The basic theoretical framework underlying our gauge theory of strings is the Carathéodory formulation of the Hamilton-Jacobi least action principle. The main features of that approach to classical dynamics can be summarized as follows:

- the variational principle takes into account a whole *family of extremals* rather than a single classical solution of the equations of motion;

- there is a remarkable duality between extremals and wave-fronts, in the sense that the classical motion of the system can be equivalently described either in terms of extremal trajectories, or in terms of propagation of the associated wave-fronts;

- the essential dynamical variable is the *slope field*. This field assigns a tangent plane to each point of the classical extremal;

- the whole theory can be elegantly formulated in terms of differential geometry.

The primary purpose of this section is to introduce the basic idea of the gauge model of strings and to set up the notation. We shall do so by discussing first the case of a relativistic point particle. This case probably represents the simplest example of a reparametrization invariant system which is rich enough to illustrate the effectiveness and simplicity of the method.

Except for a proportionality constant, the canonical action for a relativistic object is the proper area of the world-manifold spanned by the object during its evolution. Thus, the action corresponding to any extremal timelike world-hypersurface with a given initial and final configuration is nonzero. That is true for a point–particle as well.

The world–line swept by a pointlike object of rest mass m is determined by demanding that the functional

$$I[\gamma] = \int_{0,\gamma}^{T} d\lambda \, L(x, \dot{x}; \lambda) \ , \quad L(x, \dot{x}; \lambda) = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu} \ , \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda} \tag{2.1}$$

be stationary under small variations of the path γ connecting the initial and final points $x_0^\mu = x^\mu(0)$ and $x^\mu = x^\mu(T)$.

The extremal trajectory is a solution of the Euler-Lagrange equation

$$\frac{dP_\mu}{d\lambda} = 0 \ , \tag{2.2}$$

where the particle four-momentum is defined as follows

$$P_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = m\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \ , \tag{2.3}$$

and, because of reparametrization invariance, is subject to the constraint

$$-P_\mu P^\mu = m^2 \ . \tag{2.4}$$

Equations (2.2),(2.4) can be immediately solved, and give:

$$P_\mu = m\frac{v_\mu}{\sqrt{-v^2}} \ , \tag{2.5}$$

where v_μ is a constant vector field. The classical solution (2.5) of the Lagrange equations (2.2), under the constraint (2.4), introduces v^μ as the *slope field*, so-called because once eq.(2.5) is re-inserted into equation (2.3), it matches the tangent vector along the particle world–line. Indeed, once the boundary conditions are assigned, and the slope field is obtained from the Lagrange equations, one obtains through equation (2.3) an extremal curve solving the given dynamical problem. In the free particle case the slope field is constant and the world-line is a geodesic

$$\begin{aligned} x^\mu(\lambda) &= \frac{(x - x_0)^\mu}{T}\lambda + x_0^\mu \ , \\ P_\mu &= m\frac{(x - x_0)_\mu}{\sqrt{-(x - x_0)^2}} \ . \end{aligned} \tag{2.6}$$

Accordingly, the tangent vector field is constant and so is $\dot{x}^\mu(\lambda) = (x - x_0)^\mu / T$. But this is only because the particle is free. In the more general case in which external forces are present, both the four-velocity and momentum are restricted to vary along the extremal, i.e. $\dot{x}^\mu = \dot{x}^\mu(\lambda)$, $\dot{x}_\mu = P_\mu(\lambda)$. However, the Hamilton-Jacobi approach enables one to extend the four momentum to a *four* dimensional sub-manifold of Minkowski space $M^{4)}$ by taking into account that eqs.(2.6) describe a *family* of classical solutions parametrized by the initial particle position and momentum. For later convenience, we are interested in the initial spatial position of the particle defined by the three-vector $\vec{x}_0$. Then, any point x^μ belonging to a classical extremal curve, will become a function of four new " coordinates "

$$x^\mu = x^\mu(y) \ , \qquad y^\alpha \equiv (\lambda, \vec{x}_0) \ . \tag{2.7}$$

Notice that varying λ amounts to move x along one definite extremal, while variation of $\vec{x}_0$ corresponds to select a different path within the family of classical extremals. In this respect $\partial / \partial x_0^k$ represent " orthogonal directions " with respect to $\partial / \partial \lambda$. If the mapping between x^μ and y^α is one-to-one, then the classical solution (2.7) represents an invertible change of coordinates. Hence, when a family of classical solutions is properly taken into account, it is possible to define a corresponding momentum field $P_\mu(x)$, or velocity field $v^\mu(x)$, having support over a four dimensional spacetime region. The Carathéodory approach to the calculus of variations provides a well defined procedure to construct these " extended " fields, and is free of integrability problems.

One of the most interesting features of the Hamilton-Jacobi theory is to associate *wave fronts* to pointlike objects. To introduce these new geometrical objects we recall that the Jacobi action, or dynamical phase, is obtained by evaluating the line integral (2.1) along the classical solution (2.6).

$$I(x; x_0) = \int_0^T d\lambda \, L(x(\lambda), \dot{x}(\lambda)) = -m\sqrt{-(x - x_0)^2} \equiv S(x) \ , \tag{2.8}$$

Hence S defined above, becomes an ordinary function of the geodesic end- points: x which is to be understood as a variable and x_0 which should be interpreted as a parameter fixed by the initial conditions. Comparing eqs.(2.6) and (2.8), one sees that $S(x)$ must satisfy the relativistic Hamilton-Jacobi equations

$$\partial_\mu S \partial^\mu S = -m^2 \ , \tag{2.9}$$

$$\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}} = \frac{1}{m} \partial^\mu S(x) \ , \tag{2.10}$$

where, $\partial_\mu = \partial / \partial x^\mu$. The Hamilton-Jacobi wave fronts are now defined as the three-surfaces of constant dynamical phase, i.e.

$$S(x) = \mathit{const.} \tag{2.11}$$

The surfaces (2.11) obey the wave type evolution equation (2.9), and are transverse with respect to the congruence of spacetime curves solving eq.(2.10). In this sense a given extremal can be " embedded " into a system of wave fronts, any point on a constant phase surface being the intersection of one, and only one, particle world line. Then, once the family of

Hamilton–Jacobi wave fronts is known, the gradient of $S(x; x_0)$ determines a slope field in the local sense, i.e. defined at each point over the surfaces of the family. Such a slope field determines a family of particle world-lines, crossing the Hamilton–Jacobi wavefronts. The geometric object formed by the family of extremals and the associated wave fronts constitutes what is called the *complete Carathéodoryfigure*, and provides a unique characterization of the solution in Carathéodory's own version of the Hamilton-Jacobi variational principle.

It is worth emphasizing, parenthetically, that equation (2.9) was obtained after solving the equation of motion, which in the case of a point particle is very simple. However, in order to deal with the string case we need an algorithm that enables us to recover equation (2.9) without knowing the explicit form of the solution. This can be done by considering the hamiltonian form of the action integral:

$$I[\gamma] = \int_0^T d\lambda \left[P_\mu \, \dot{x}^\mu - N(\lambda)(P^2 + m^2) \right] \ , \tag{2.12}$$

where, $N(\lambda)$ is a lagrange multiplier enforcing the constraint (2.4). If we evaluate $S(x)$ on a classical trajectory, say $x = y(\lambda)$, then we find

$$I(x, x_0) = \int_{x_0}^{x} P_\mu(y) \, dy^\mu = \int_0^T d\lambda \, L \left(y(\lambda), \dot{y}(\lambda) \right) \ . \tag{2.13}$$

Since $S(x)$ is a function of the extremal end-points only, then, when the path is varied, the requirement that $I(x, x_0)$ be an extremum, leads to:

$$\int_0^T d\lambda \, \partial_{[\,\mu} P_{\nu]} \, \dot{y}^\nu \, \delta y^\mu + \left[P_\mu \, \delta y^\mu \right]_{x_0}^{x} = 0 \ . \tag{2.14}$$

Here, the variation of the trajectory end-points is independent of the path deformation. Then, setting $\delta y^\mu(T) = dx^\mu$, and $\delta y^\mu(0) = 0$, we obtain

$$\partial_{[\,\mu} P_{\nu]} \frac{dy^\nu}{d\lambda} = 0 \ , \tag{2.15}$$

$$\frac{\partial S}{\partial x^\mu} = P_\mu(x) \ . \tag{2.16}$$

The above set of equations tells us that along a classical trajectory the linear momentum can be expressed as the gradient of a scalar function. This is exactly what we did in equations (2.10, 2.9).

This result, simple as it is in this case, can be translated in the language of differential geometry as follows: the momentum one-form $\theta(x) \equiv P_\mu(x) dx^\mu$ is closed along a classical trajectory so that, at least locally, it is an exact one-form

$$d\theta \left(x = x(\lambda) \right) = 0 \ \rightarrow \theta \left(x = x(\lambda) \right) = dS \left(x = x(\lambda) \right) = L \left(x(\lambda), \dot{x}(\lambda) \right) d\lambda \ . \tag{2.17}$$

For reparametrization-invariant theories, the last equality can be written as:

$$\theta(x = x(\lambda)) = \dot{x}^\mu(\lambda) \partial_\mu S(x) d\lambda \ . \tag{2.18}$$

In this form, the above results apply to the string case as well.

B. The Carathéodoryleast action principle

The ideas expressed above in terms of the one-form $\theta(x)$, find an immediate application in the Carathéodory formulation of the Hamilton-Jacobi theory. The main point is to modify the lagrangian according to:

$$L(\dot{x}) \longrightarrow L^*(\dot{x}) = L(\dot{x}) - \frac{dS(x)}{d\lambda} = L(\dot{x}) - \dot{x}^\mu \partial_\mu S(x). \tag{2.19}$$

As it is well known, the addition of a total derivative does not affect the equations of motion and therefore L and L^* are dynamically equivalent. The principle of least action *applied to the new lagrangian* yields:

$$\begin{aligned} 0 = \delta I^*(x, x_0) &= \int_0^T d\lambda\, L^*(\dot{x}) = \\ &= \int_0^T d\lambda\, \delta x^\mu(\lambda) \left[\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} \right] + \\ &+ \left[\delta x^\mu(\lambda) \left(\frac{\partial L}{\partial \dot{x}^\mu} - \partial_\mu S(x) \right) \right]_0^T . \end{aligned} \tag{2.20}$$

As a matter of fact, even if *the variation is restricted to a family of solutions of the Lagrange equations*, with the boundary condition $\delta x^\mu(0) = 0$, we still have to impose

$$\delta I^* \Big|_{x=x(\lambda)} = 0 \rightarrow P_\mu(x) = \partial_\mu S(x) \ . \tag{2.21}$$

i.e., the requirement that the momentum is the gradient of the Jacobi phase. From the definitions (2.19), (2.21), it follows that the solutions of the above variational problem can be obtained from the condition

$$\left(\frac{\partial L^*}{\partial \dot{x}^\mu} \right)_{x=x(\lambda)} = 0 \ . \tag{2.22}$$

The proper length lagrangian is invariant under reparametrization, therefore it is homogeneous of degree one with respect to the generalized velocity $\dot{x}^\mu$, so that

$$L^*(x(\lambda), \dot{x}(\lambda)) = 0 \tag{2.23}$$

follows from equation (2.22). For a more general non- reparametrization invariant theory, eqs.(2.22) and (2.23) represent the *fundamental equations of the calculus of variation,* the associated complete figure consisting of a three-parameters congruence of extremals intersecting a one-parameter family of wave-fronts. The usefulness of this modified variational approach and its connection with field theory, is appreciated when one considers not a single particle but a *statistical ensemble* of them.

III. CARATHÉODORY STRING ACTION

A. Hamilton-Jacobi theory for the closed string

A closed string is formally described by a two dimensional domain D in the space of the parameters $\xi^a = (\tau, \sigma)$ which represent local coordinates on the string world-sheet, and by an embedding Ω of D in Minkowski spacetime M, that is $\Omega : \xi \in D \to \Omega(\xi) = X^\mu(\xi) \in M$. We assume that the domain D is simply connected and bounded by a *spacelike* curve $\gamma : \xi^a = \xi^a(s)$. Furthermore, we can choose the parameter s to vary in the range $0 \leq s \leq 1$. Then, Ω maps the boundary γ into the *spacelike loop* C in spacetime: $x^\mu = x^\mu(\xi(s)) = x^\mu(s)$; $x^\mu(s+1) = x^\mu(s)$. The resulting picture describes a spacetime world-sheet whose only " free end-point " is the spacelike loop $x = x(s)$. The Euler-Lagrange equations, derived from

$$L(x, \dot{x}; \xi) = -m^2 \sqrt{-\gamma(\xi)} \ , \quad \gamma(\xi) = \frac{1}{2} \dot{x}^{\mu\nu} \dot{x}_{\mu\nu} \ ,$$
$$\dot{x}^{\mu\nu} = \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu \ , \qquad m^2 = \frac{1}{2\pi\alpha'} \ , \tag{3.1}$$

read

$$\epsilon^{ab} \partial_a \Pi_{\mu\nu} \partial_b x^\nu = 0 \ , \qquad \Pi_{\mu\nu} \equiv \frac{\partial L}{\partial \dot{x}^{\mu\nu}} = m^2 \frac{\dot{x}_{\mu\nu}}{\sqrt{-\gamma(\xi)}} \ , \tag{3.2}$$

and, the *area momentum* $\Pi_{\mu\nu}$, canonically conjugate to $\dot{x}^{\mu\nu}$, satisfy the "mass-shell" constraint

$$-\frac{1}{2} \Pi_{\mu\nu} \Pi^{\mu\nu} = m^4 \tag{3.3}$$

because of the reparametrization invariance of the theory. The two equations (3.2, 3.3) define the string slope field $\Phi_{\mu\nu}(\xi)$ as

$$\Pi_{\mu\nu}(\xi) = m^2 \frac{\Phi_{\mu\nu}(\xi)}{\sqrt{-|\Phi|^2}} \ , \qquad |\Phi|^2 \equiv \frac{1}{2} \Phi_{\mu\nu}(\xi) \Phi^{\mu\nu}(\xi) \tag{3.4}$$

where,

$$\epsilon^{ab} \partial_a \frac{\Phi_{\mu\nu}(\xi)}{\sqrt{-|\Phi|^2}} \partial_b x^\nu = 0 \ . \tag{3.5}$$

Dealing with a spatially extended object means that the slope field is now a bi-vector field, which assigns to every point a plane. Once such a field of planes is known, the extremal surface swept by the string evolution is given by the equation

$$\epsilon^{ab} \partial_a x^\mu \partial_b x^\nu = \Phi^{\mu\nu}(\xi) \ , \tag{3.6}$$

which represents the condition that, at each point, the world-sheet tangent element overlaps with the corresponding tangent plane. Equation (3.6) suggests a hydrodynamic picture of

the string as a vortex line co-moving in a viscous fluid described by the velocity field $\Phi^{\mu\nu}(\xi)$. In this picture, the matching equation (3.6) represents the no–slipping condition for the string within the fluid.

To obtain the Hamilton–Jacobi equations for the string, we turn to the hamiltonian form of the Nambu-Goto action

$$I[X(\xi)] = \frac{1}{2}\int_{X(\xi)} dx^\mu \wedge dx^\nu \Pi_{\mu\nu}(x) - \int_D d^2\xi\, N(\xi)\left[\frac{1}{2}\Pi_{\mu\nu}\Pi^{\mu\nu} + m^4\right] , \tag{3.7}$$

and consider the variation of I among the classical solutions of equation (3.6). Thus, I becomes a functional of the boundary C alone, the constraint (3.3) is satisfied, and we find

$$\delta I[X(\xi)] \to \delta I[C] = -\int_D d^2\xi \partial_{[\lambda}\Pi_{\mu\nu]}\dot{x}^{\lambda\mu}\delta x^\nu + \oint_C \Pi_{\mu\nu} dx^\mu \delta x^\nu . \tag{3.8}$$

The first term in (3.8) must vanish, otherwise a dependence on the whole string world-sheet, and not on its boundary alone, would be implied. The second term represents the response of the action to a variation of the boundary. Then,

$$\partial_{[\lambda}\Pi_{\mu\nu]}\,\dot{x}^{\lambda\mu} = 0 , \tag{3.9}$$

$$P_\mu(s) \equiv \frac{1}{\sqrt{x'^2}}\frac{\delta I[C]}{\delta x^\mu(s)} = \Pi_{\mu\nu}(x(s))\frac{x'^\nu}{\sqrt{x'^2}} . \tag{3.10}$$

Equation (3.9) is nothing but the Bianchi identity for $\Pi_{\mu\nu}$ projected over an extremal surface, and tells us that strings are somehow related to gauge potentials. *Here is where one encounters the problem of " thickening " the world sheet* [15]: it would be tempting to conclude that (3.9) is satisfied not only on the string world-sheet but over a *spacetime region*, in which case one would define, $\Pi_{\mu\nu}(x) = \partial_{[\mu}A_{\nu]}(x)$. However, this conclusion is too hasty because for a two-surface, $\det \dot{x}^{\mu\nu} = 0$ [13], and (3.9) admits solutions for which $\partial_{[\lambda}\Pi_{\mu\nu]} \neq 0$ as well. The vanishing of $\det\dot{x}^{\mu\nu}$ is due to the low dimensionality of the string world-surface: in two dimensions there is no totally anti-symmetric three index tensor. To circumvent this difficulty, we have to take into account a two-parameter congruence of surfaces, $x^\mu = x^\mu(\tau, \sigma, \theta_1, \theta_2)$, rather than a single extremal world-sheet, and treat the labels θ_1, θ_2 as additional coordinates. Then, $\det\dot{x}^{\mu\nu}$ becomes a function of four coordinates, instead of two, and is generally different from zero. The congruence of extremal world-surfaces can be interpreted as a mapping from the extended parameter space to the physical spacetime. Then, the two-form

$$\theta(x) = \frac{1}{2}\Pi_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu , \tag{3.11}$$

is closed over a two-parameter family of classical solutions $x^\mu = x^\mu(\xi^a, \theta^i)$, and, at least locally, can be written in terms of a gauge potential $A_\mu(x)$

$$d\theta\,(x = x(\xi, \theta)) = 0 , \to \Pi_{\mu\nu}(x) = \partial_{[\mu}A_{\nu]}(x) . \tag{3.12}$$

In other words, the action for a family of classical solutions, i.e.

$$I[C] = \frac{1}{2}\int_{X(\xi)} dx^\mu \wedge dx^\nu\, \Pi_{\mu\nu}(x) , \tag{3.13}$$

is a boundary functional only if $\Pi_{\mu\nu}$ can be written as the curl of a vector potential, in which case $I[C]$ becomes a loop integral

$$I[C] = \oint_C dx^\mu A_\mu = \int_D d^2\xi L\left(x(\xi), \dot{x}(\xi)\right) \ . \tag{3.14}$$

Now, we still have the freedom to fix the rank[1] of $\theta(x)$. In four dimensions a 2-form has either rank-four or rank-two, which means that θ can be written in terms of four or two independent one-forms. The rank-four option means

$$\theta(x) = dS^1 \wedge dT^1 + dS^2 \wedge dT^2. \tag{3.15}$$

But, in this case the wavefronts degenerate into points, so that the complete Carathéodoryfigure associated with the string motion becomes meaningless. A first attempt to solve this problem was proposed by Nambu [10], who tried to generalize the Hamilton–Jacobi formalism for non-relativistic system to the string case as well. The starting point is the non-reparametrization invariant Schild action [16] which leads to a six-dimensional Hamilton–Jacobi formalism where $S^{i)} = S^{i)}(x,\xi)$, and $T^{i)} = T^{i)}(x,\xi)$, $i = 1, 2$. However, this approach only apparently solves the integrability problem. In fact, the resulting wavefronts are two-dimensional surfaces, while the general rule for mechanical systems of this kind would require the dimension of the wavefronts to be four [17].

The alternative possibility, that we consider, is to preserve reparametrization invariance and change the lowest rank for θ.

By definition, a wavefront is the integral sub-manifold corresponding to the characteristic subspace $C(\theta)$. A vector field Y belongs to $C(\theta)$ if it annihilates both θ and $d\theta$, i.e. $i(Y)\theta = 0$, $i(Y)d\theta = 0$. According to the general definition above, the wavefront will be $(D-c)$–dimensional where D is the spacetime dimension and $c = dim\, C(\theta)$ is the *class* of θ [18]. If $d\theta = 0$, $\rightarrow c = r = rank(\theta)$, and the wavefronts are $(D-r)$– dimensional hypersurfaces. Hence, by choosing for θ the closed, rank–two, two–form

$$\theta(x) = dS \wedge dU \ , \tag{3.16}$$

$$\begin{aligned}\theta\left(x = x(\xi)\right) &= \dot{x}^{\mu\nu}\partial_\mu S \partial_\nu U \, d^2\xi \\ &= L\left(x(\xi), \dot{x}(\xi)\right) \, d^2\xi\end{aligned} \tag{3.17}$$

the class c=rank=2 and the wave-fronts are $D - r$ =two-dimensional, integral manifolds, given as solution of the equations

$$S(x) = const. \equiv \sigma_1 \ , \tag{3.18}$$

$$U(x) = const. \equiv \sigma_2 \ . \tag{3.19}$$

The surfaces defined by (3.18,3.19) satisfy the "wave equation"

---

[1] We recall that the rank r of a p-form ω is the minimal number r of linearly independent 1-forms in terms of which ω can be expressed.

$$\frac{1}{2}\partial_{[\mu}S\partial_{\nu]}U\partial^{[\mu}S\partial^{\nu]}U = -m^4 \ , \tag{3.20}$$

and, together with the family of extremal surfaces solving

$$\frac{\dot{x}^{\mu\nu}}{\sqrt{-\gamma(\xi)}} = \frac{1}{m^2}\partial^{[\mu}S\partial^{\nu]}U \ , \tag{3.21}$$

form the complete Charathèodory figure associated with the string motion.

Turning now to the boundary eq.(3.10), we note that it provides the Jacobi definition of the string momentum $P_\mu(s)$ as the dynamical variable canonically conjugate to the world-sheet boundary deformation. By inverting equation (3.10) we can give meaning to the area momentum on the boundary

$$\Pi_{\mu\nu}(s) = \frac{P_{[\mu}(s)x'_{\nu]}(s)}{\sqrt{x'^2}} \ , \tag{3.22}$$

where we have taken into account the orthogonality relation $P_\mu(s)x'^\mu = 0$. Equation (3.22) shows that the string momentum satisfies the mass-shell relation

$$\left(\int_0^1 ds\,\sqrt{x'^2}\right)^{-1}\int_0^1 ds\,\sqrt{x'^2}\,P_\mu(s)P^\mu(s) = -m^4 \ , \tag{3.23}$$

following from equation (3.3), and an average over the loop parameter s which makes eq.(3.23) reparametrization invariant. The relation between classical momenta will be a key ingredient in the quantum formulation of string dynamics through the Correspondence Principle.

B. The averaged Charathéodory variational principle

The Carathéodory formulation of the HJ variational principle is obtained from the modified action functional:

$$\begin{aligned} I[X(\xi)] &= \int_D d^2\xi\, L(\dot{X}) - \oint_C S(x)\, dU(x) \ , \\ &= \int_D d^2\xi\, \left[L(\dot{X}) - \frac{1}{2}\dot{x}^{\mu\nu}\partial_{[\mu}S\partial_{\nu]}U\right] \ , \\ &= \int_D d^2\xi\, L^*(\dot{X}) \ . \end{aligned} \tag{3.24}$$

As far as the variational problem is concerned, the presence of an additional boundary term is irrelevant, and the classical solutions will be the same ones that follow from the Nambu-Goto action. However, the presence of the slope field in Carathéodory's formulation of the least action principle, is instrumental in connecting the description of strings dynamics in terms of mechanical variables with the formulation in terms of a gauge potential. To appreciate this point which is essential in the postmodern gauge interpretation of string theory, observe that the slope field which minimizes the action (3.24) must be a solution of the equation

$$\frac{\partial L^*(x,\dot{x};\xi)}{\partial \dot{x}^{\mu\nu}}\Big|_{x=x(\xi)} = 0 \ , \rightarrow \Pi_{\mu\nu}(\xi) = \partial_{[\mu}S\partial_{\nu]}U \ , \rightarrow L^*(x,\dot{x};\xi)\Big|_{x=x(\xi)} = 0 \ . \tag{3.25}$$

Equation (3.25) requires $S(x)$ and $U(x)$ to be solutions of the Hamilton–Jacobi equation (3.20) and the family of extremal surfaces $x = x(\xi)$ to solve equation (3.20). Thus, the condition (3.25) defines the same complete Carathéodory figure associated with the string motion as the hamiltonian action (3.7), and is therefore equivalent to it. The novel feature of the string action *a–la Carathéodory* is the presence of an "interaction term" between the loop current $j^\mu(x;C) = \int_0^1 ds\, x'^\mu\, \delta^4[x - x(s)]$, and the " external electromagnetic gauge potential " $A_\mu = S\partial_\mu U (= -U\partial_\mu S + \partial_\mu(SU))$. With this interaction term the model becomes invariant under the gauge transformation $\delta A_\mu = \partial_\mu \Lambda$, and, furthermore, the corresponding " electromagnetic strength " $F_{\mu\nu}$ has minimal *rank-two* [18], which is the distinctive feature of a string-like field excitation. All this suggests that it should be possible to give a gauge form to the Nambu-Goto action as well. This objective is achieved by considering a statistical ensemble of strings, in other words, we switch from a single string evolution problem to the dynamics of a "Gibbs ensemble" of strings. For such a purpose we need:

i) a *dense* set $\mathcal{P}$ of spacetime surfaces $x^\mu = x^\mu(\xi)$, i.e., a family of extremals covering at least a four-dimensional region $G^{4)} \subset M^{4)}$. We will refer to it as the "path-space of the string". After imposing a particular variational principle, $\mathcal{P}$ will become a subset of all the possible world-sheets swept by the string evolution;

ii) an invariant measure given over this set assigning the probability for the string to move along a given trajectory. i) The path-space $\mathcal{P}$ can be constructed as follows.

Consider the above mentioned family of non-intersecting surfaces: since the surfaces are two-dimensional objects, and since they are required to form a dense set covering a four-dimensional submanifold, it will consist of a *two parameter*, say u^1, u^2, family of surfaces. The rationale for introducing $\mathcal{P}$ is to set up a one–to–one correspondence between (x^0, x^1, x^2, x^3) in $G^{4)}$ and $(\sigma, \tau; u^1, u^2)$ in parameter space. Still, their "form" remains free, i.e. $G^{4)}$ can be covered by planes, or by spherical shells or whatever. As a characterization of the whole family, take, e.g., the tangent elements $\dot{x}^{\mu\nu}(\xi)$ to every surface, and require them to satisfy:

$$m^2 \frac{\dot{x}_{\mu\nu}(\xi)}{\sqrt{-\gamma(\xi)}} = \partial_{[\mu}S\partial_{\nu]}U \ . \tag{3.26}$$

For the moment, $S(x)$ and $U(x)$ are two *generic* functions: they shall become the two Jacobi potentials of the H-J theory only after certain conditions are met. ii) As an invariant measure (under reparametrization of the surfaces belonging to $\mathcal{P}$), introduce a positive definite weight $\mu(u)$ defined on the space of parameters u^1, u^2. Any pair u^1, u^2 will label a single surface within $\mathcal{P}$. Then, we can define the probability for a string to move along a sample path with parameters in the subset $\{B\}$ in parameter space, as:

$$prob(B) \equiv \int_B d^2u\ \mu(u) \ . \tag{3.27}$$

The peaks of $\mu(u)$ correspond to more probable values of the parameters, i.e., to statistically preferred classical paths. Thus, even if the evolution remains strictly deterministic, a random

element is introduced into the model. At this stage the density $\mu(u)$ is an arbitrarily assigned positive definite function. However, in the last part of the paper, we will show how this distribution is fundamentally related to the quantum dynamics of the system.

Given the action for a two-parameter family of surfaces randomly distributed in the parameter space, the slope field solving the least action principle is obtained by averaging the Carathéodory action over the parameters u^a

$$\delta\langle I\rangle = \delta \int d^2u \; \mu(u) \int_D d^2\xi \, L^* \left(\dot{x}(\xi; u)\right) = 0. \tag{3.28}$$

Since $\mu(u)$ does not depend on ξ^a and is positive definite, eq. (3.28) is just $\int$ eq.(3.24) $\mu(u)\, d^2u$, and gives the usual equations of motion upon variation of the appropriate variables. This means that, imposing the " average minimum action principle ", the family of surfaces in $\mathcal{P}$ becomes the congruence of extremals of the Carathéodory complete figure, and $S(x)$, $U(x)$ become the two Jacobi phases.

The reason for integrating over d^2u becomes clear as soon as we transform the variables $(\xi; u)$ to the x^μ. Eq.(3.28) then reads:

$$\langle I\rangle = \int_{G^{4)}} d^4x \, \mu\left(u(x)\right) \, Z(x) \, L^* \left(\dot{x}^{\mu\nu}(x)\right) = \int_{G^{4)}} d^4x \, L^* \left(\rho(x)\dot{x}^{\mu\nu}(x)\right) \ , \tag{3.29}$$

where $\dot{x}^{\mu\nu}(x) \equiv \dot{x}^{\mu\nu}(\xi(x), u(x))$, and $Z(x) \equiv \partial(\xi^1, \xi^2; u^1, u^2)/\partial(x^0, x^1, x^2, x^3)$. Moreover, in view of the homogeneity of the lagrangian, $\rho(x) \equiv Z(x)\mu\left(u(x)\right)$ has been taken into the argument of L^*. Finally, introducing the " Gibbs ensemble associated current "

$$W^{\mu\nu}(x) \equiv \mu(x)Z(x)\dot{x}^{\mu\nu}(x) \equiv \rho(x)\dot{x}^{\mu\nu}(x) \ , \tag{3.30}$$

we can write $\langle I\rangle$ as the action for a relativistic field theory:

$$I_{field} = \langle I\rangle = \int d^4x \left[-m^2 \sqrt{-\frac{1}{2} W^{\mu\nu}(x) W_{\mu\nu}(x)} - \frac{1}{2} W^{\mu\nu}(x) \partial_{[\,\mu} A_{\nu]} \right] \ . \tag{3.31}$$

This action was introduced by the authors [13] as a gauge model with a *single string* slope-field $W^{\mu\nu}$ proportional to the classical string current, i.e.

$$W^{\mu\nu} = const. \int_D d^2\xi \, \delta^4(x - x(\xi)) \dot{x}^{\mu\nu}(\xi) \tag{3.32}$$

which represents a special solution of the classical field equations

$$m^2 \frac{W_{\mu\nu}(x)}{\sqrt{-\frac{1}{2} W^{\rho\sigma}(x) W_{\rho\sigma}(x)}} = \partial_{[\mu} A_{\nu]}(x) \ , \tag{3.33}$$

$$\partial_\mu W^{\mu\nu}(x) = 0 \ . \tag{3.34}$$

However, within the stochastic approach advocated here, a more general solution can be given. Indeed, with the definition $\overline{\rho}(x) \equiv m^2 \sqrt{-\gamma}\, \rho(x)$, it is immediate to check that

$W^{\mu\nu}(x) = (\overline{\rho}(x)/m^4)F^{\mu\nu}(x)$ is a solution of eq. (3.33), (3.34) if $F^{\mu\nu}(x)$ satisfies the H–J equation

$$-\frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x) + m^4 = 0 \ , \tag{3.35}$$

and the " current " $\overline{\rho}(x)\, F^{\mu\nu}$ is divergenceless

$$\partial_\mu \left(\overline{\rho}(x)\, F^{\mu\nu}(x)\right) = 0 \ . \tag{3.36}$$

Rather than the slope-field associated to the single string case, $W^{\mu\nu}(x)$ represents, now, the divergence–free Gibbs current for a statistical system of filamentary structures. One can visualize this system as a *fluid of relativistic strings* covering the spacetime manifold with a distribution density $\overline{\rho}(x)$ and a velocity field $\sim F^{\mu\nu}$. In this *hydrodynamic* interpretation of the action (3.31), two very general questions come immediately to mind. The first pertains to the classical domain, the second to the quantum domain: if Einstein's theory of gravitation is any guide, one would expect that the geometry of spacetime is shaped by the matter–string distribution. We would like to determine how. In other words, is the classical spacetime manifold purely Riemannian?, *and* is there a discernible structure in spacetime? We shall discuss this intriguing aspect of our theory in Sect.V . Presently, we turn to the second fundamental issue, namely the construction of a quantum theory of strings based on the statistical classical formulation developed so far. In the next section we will build this theory from the " ground " up, in a literal sense, by interpreting the fluid of relativistic strings as a model for the *vacuum* in a stochastic quantum theory of strings.

IV. SEMI-CLASSICAL LIMIT OF THE QUANTUM LOOP EQUATION

A. The string functional

The primary objective of this section is to show how the classical equations (3.36) emerge as the $\hbar \to 0$ limit of the " string wave equation "

$$\left[\frac{\hbar^2}{2}\left(\int_0^1 ds\,\sqrt{x'^{\,2}}\right)^{-1}\int_0^1 ds\,\sqrt{x'^{\,2}}\,\frac{\delta^2}{\delta\sigma^{\mu\nu}(s)\delta\sigma_{\mu\nu}(s)} - m^4\right]\Psi[C] = 0 \ , \qquad m^2 \equiv 1/2\pi\alpha' \ . \tag{4.1}$$

Here, $\Psi[C]$ denotes a *functional* of the spatial loop $C : x^\mu = x^\mu(s)$ describing the string, and $\delta/\delta\sigma^{\mu\nu}(s)$, denotes the functional derivative with respect to the area element. Eq.(4.1) is nothing but the string momentum mass shell condition (3.23) expressed through the quantum operators canonically conjugated to $P_\mu(s)$ and $\Pi_{\mu\nu}(s)$ through the Correspondence Principle:

$$P_\mu(s) \to i\hbar\frac{1}{\sqrt{x'^{\,2}}}\frac{\delta}{\delta x^\mu(s)} \ , \tag{4.2}$$

$$\Pi_{\mu\nu}(s) \to i\hbar\frac{\delta}{\delta\sigma^{\mu\nu}(s)} \ , \tag{4.3}$$

and

$$\frac{\delta}{\delta x^\mu(s)} = x'^\nu \frac{\delta}{\delta \sigma^{\mu\nu}(s)} \ . \tag{4.4}$$

To simplify the notation, one can choose as loop parameter, s, the proper length defined as $ds^2 = dx^\mu dx_\mu$, so that $x'^2 = 1$. Then, eq.(4.1) takes the form given by Hosotani [12].

Before going into technical details, it may be useful to clarify in which sense we " quantize " the string. As a matter of fact, when dealing with extended objects, there are at least two conceptually different ways to interpret the meaning of the term quantization. The first and most used approach to quantization treats the string as a mechanical system, quantizing its small oscillations by canonical or path integral methods and interpreting the resulting excitation states of the string as particles of different mass and spin. Parallel to this approach, there is the much less investigated geometrical framework, which we wish to explore here, where the string is not merely a mechanical device to generate a particle spectrum, but is regarded as a physical object in itself whose shape can quantum mechanically fluctuate between different configurations. From this viewpoint the wave equation (4.1) is the string counterpart of the Wheeler-DeWitt equation, in the sense that it determines the quantum geometry of the object under investigation, i.e. the string *spatial configuration* in our case, and the *spatial metric* in the canonical formulation of quantum gravity. *From our own vantage point, to quantize the string means to determine the probability amplitude for the string to have a spatial shape described by the loop C.* At this point, two objections to this interpretation can be immediately raised: first, the functional $\Psi[C]$ is defined in an abstract loop-space, while one would like to define a probability amplitude, or density, to find a string with a given shape in the physical spacetime; second, even if we succeed in connecting quantities defined in loop-space to their spacetime counterpart, we expect to be unable to introduce a positive definite probability density because of the relativistic invariance of the system. While the first problem can be solved, at least formally, as we shall see in the rest of this section, the second difficulty finds an acceptable solution only in a " second quantization " scheme, in which $\Psi[C]$ plays the role of field operator, creating and destroying loops of a given shape. We shall come back to this point at the end of this section. Meanwhile we shall still refer to $\Psi[C]$ as the " probability amplitude " to underscore the fact that we are dealing with a first quantized theory in the semi–classical limit. We believe that this approach not only is a useful introduction to the second quantization of the string functional, but is also a necessary intermediate step to bridge the gap between the quantum equation (4.1) and the classical equations (3.33), (3.36). In this connection, we should mention Hosotani's *loop variable* ansatz [12]

$$\Psi[C] \sim \exp\left(\frac{i}{\hbar}\oint_C A_\mu(x) dx^\mu\right) \ . \tag{4.5}$$

This wave functional does serve the purpose of relating equation (4.1) with the classical Hamilton–Jacobi equations for the string, but corresponds to a "string plane wave" , i.e. to an unlocalized state in loop space that assigns an equal probability amplitude to any loop shape. In the stochastic quantum theory of strings that we have in mind, we are more interested in a new family of states for which different geometric configurations are weighted by a non trivial probability amplitude. This is by no means a settled issue and therefore, without pretence of mathematical rigor, we proceed simply by analogy with ordinary quantum mechanics, bearing in mind that the real test of consistency of our approach is to derive

the classical Hamilton-Jacobi equations for the string starting from the wave equation for the string functional. To this end, and in order to assign a probabilistic interpretation to $\Psi[C]$, which has canonical dimensions of a $(length)^2$, we proceed in two steps: first we must introduce a suitable length scale in the normalization condition for $|\Psi[C]|^2$. At this stage the only length scale we have at our disposal is provided by the string tension $m^2 = 1/2\pi\alpha'$, and therefore we define

$$m^4 \int D[C]\, |\Psi[C]|^2 = 1. \tag{4.6}$$

Next, we define the probability density (in Minkowski spacetime) by inserting the identity

$$\int d^4x\, \delta\left[x - x(s)\right] = 1 \ , \tag{4.7}$$

into the l.h.s. of eq.(4.6):

$$\int d^4x\, \overline{\rho}(x) = 1 \ \rightarrow \overline{\rho}(x) \equiv m^4 \int D[C]\, \delta^{4)}\left[x - x(s)\right]\ |\Psi[C]|^2 \ . \tag{4.8}$$

According to this definition, the function $\overline{\rho}(x)$ represents the (probability) density of loops of arbitrary shape within an elementary volume element centered at the point x. Each different shape is weighted by the factor $|\Psi[C]|^2$ and equation (4.8) provides the correct unit normalization for the total probability density. With hindsight, we have denoted the quantum density by the same symbol used for its classical counterpart. The reason for that will become clear shortly.

The next step in our stochastic approach, which we describe in the next subsection, is to show that the same probability density enters in the semi-classical form of the regularized probability current derived from the WKB string functional

$$\Psi[C] \equiv A[C] \exp \frac{i}{\hbar} \oint_C dy^\mu A_\mu(y) \ . \tag{4.9}$$

This wave functional represents our own ansatz for a loop variable describing a string state of (area) momentum $\Pi_{\mu\nu}(= \partial_{[\mu} A_{\nu]})$ and amplitude $A[C]$, which we assume to be slowly varying over the loop space.

B. Point-splitting regularization

The stochastic quantum mechanical approach that we are setting up would be ill defined without solving a technical problem of regularization related to the non-local character of the loop field (4.9). Presently, our purpose is to suggest a possible computational procedure to deal with the differential operator in eq.(4.1). In fact, a naive attempt to evaluate the second functional derivative of the WKB ansatz (4.9) results in a term proportional to $\delta(0)$. This divergence can be kept under control by regularizing eq.(4.1) through point-splitting:

$$\lim_{\epsilon \to 0} \left[\frac{\hbar^2}{2} \int_0^1 ds\, \frac{\delta^2}{\delta\sigma^{\mu\nu}(s - \epsilon/2)\delta\sigma_{\mu\nu}(s + \epsilon/2)} - m^4 \right] \Psi[C] = 0 \ , \tag{4.10}$$

where the limit $\epsilon \to 0$ is taken at the end of all calculations. As long as ϵ is non-vanishing, the two points $s_\pm \equiv s \pm \epsilon/2$ can be considered as independent variables (with respect to s) and the integration over the string parameter reduces to a multiplication by a constant equal to one.

Next, we multiply eq.(4.10) by $\Psi^*[C]$ and (functionally) integrate over the loop shape, then we split the real and imaginary part of the equation thus obtained, and insert the WKB ansatz. The limit $\epsilon \to 0$ is harmless in the real part of the equation, and one finds

$$\int D[C] \int_0^1 ds A[C] \left[\frac{\hbar^2}{2} \frac{\delta^2 A[C]}{\delta\sigma^{\mu\nu}(s)\delta\sigma_{\mu\nu}(s)} - A[C] \left(\frac{1}{2} F_{\mu\nu}(x(s)) F^{\mu\nu}(x(s)) + m^4 \right) \right] = 0 \ . \quad (4.11)$$

By inserting the unity (4.7) into (4.11), we can exchange $F_{\mu\nu}(x(s))$ for $F_{\mu\nu}(x)$. Then, equation (4.11) is satisfied at any spacetime and loop space point whenever the quantity inside the square bracket is vanishing, i.e.

$$-\frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) - m^4 + \frac{\hbar^2}{2} \int_0^1 ds \frac{1}{A[C]} \frac{\delta^2 A[C]}{\delta\sigma^{\mu\nu}(s)\delta\sigma_{\mu\nu}(s)} = 0 \ , \quad (4.12)$$

which is nothing but equation (3.36) plus quantum corrections of order $\hbar^2$. To recover the equation for the divergence of the Gibbs current, we note that in the limit of vanishing ϵ:

$$Im \left\{ \int_0^1 ds\, \Psi^*[C] \frac{\delta}{\delta\sigma^{\mu\nu}(s_-)} \frac{\delta\Psi[C]}{\delta\sigma_{\mu\nu}(s_+)} \right\} \sim Im \left\{ \int_0^1 ds \frac{\delta}{\delta\sigma^{\mu\nu}(s_-)} \Psi^*[C] \frac{\delta\Psi[C]}{\delta\sigma_{\mu\nu}(s_+)} \right\} \ , \quad (4.13)$$

because the two terms containing first order derivatives cancel against each other. Taking now the limit $\epsilon \to 0$, we obtain the " continuity equation " for the current density in loop space:

$$\frac{1}{2} \int_0^1 ds \frac{\delta}{\delta\sigma^{\mu\nu}(s)} J^{\mu\nu}[C; s] = 0 \ , \quad J^{\mu\nu}[C; s] \equiv \frac{\hbar}{2i} \Psi^*[C] \frac{\overleftrightarrow{\delta}}{\delta\sigma_{\mu\nu}(s)} \Psi[C] \ . \quad (4.14)$$

This regularization procedure, artificial as it may seem, not only eliminates the offending $\delta(0)$–term, but also leads to the correct classical equations of motion. Indeed, inserting eq.(4.9) into eq.(4.14) we obtain

$$J^{\mu\nu}[C; s] = A^2[C]\, F^{\mu\nu}(x(s)) \ , \quad (4.15)$$

which is the WKB form of the loop space current density. The corresponding spacetime current density is obtained as before by integrating eq. (4.15) over all possible loop configurations through the point x and over the parameter s:

$$J^{\mu\nu}(x) \equiv \int D[C]\, \delta^{4)}[x - x(s)]\ J^{\mu\nu}[C; s] \, . \quad (4.16)$$

Inserting again our WKB functional:

$$\begin{aligned} J^{\mu\nu}(x) &= \int D[C]\, \delta^{4)}[x - x(s)]\ A^2[C]\, F^{\mu\nu}(x) \\ &= \frac{\overline{\rho}(x)}{m^4} F^{\mu\nu}(x) \ . \end{aligned} \quad (4.17)$$

The last equality in (4.17) connects the WKB current density to the classical current through the identification of the (quantum) density $\overline{\rho}(x)$, as defined in (4.8), with the classical density $\overline{\rho}(x) = m^2\sqrt{-\gamma}\, Z(x)\, \mu(x)$. Furthermore, if we require that the divergence of $J^{\mu\nu}(x)$ vanishes, then we obtain the classical equation (3.36). The general form of the quantum current in Minkowski spacetime can be obtained again summing over all the possible loop shapes through the point x. The final result is the following correspondence between classical and quantum quantities

$$\begin{array}{lcl} J^{\mu\nu}(x) = \int D[C]\, \delta^{4)}\, [x - x(s)]\ J^{\mu\nu}[C; s] & \xrightarrow{\hbar\to 0} & W^{\mu\nu}(x) = (\overline{\rho}(x)/m^4)\, F^{\mu\nu}(x) \\ \overline{\rho}(x) = m^4 \int D[C]\, \delta^{4)}\, [x - x(s)]\ |\Psi[C]|^2 & \xrightarrow{\hbar\to 0} & \overline{\rho}(x) = m^2\sqrt{-\gamma}\, Z(x)\, \mu(x) \\ \partial_\mu J^{\mu\nu}(x) = 0 & \xrightarrow{\hbar\to 0} & \partial_\mu \left(\overline{\rho}(x) F^{\mu\nu}(x)\right) = 0 \end{array} \tag{4.18}$$

In the above code of correspondence we have imposed the vanishing of the divergence of the current having in mind the analogy with the continuity equation in ordinary quantum mechanics. However, when dealing with a relativistic system, such a motivation is not justified. In our case $\overline{\rho}(x)$ does not correspond to any of the components of $J^{\mu\nu}(x)$; moreover, as in the case of the Klein-Gordon equation for the pointlike particle, a " probability " current such as $J^{\mu\nu}(x)$ is not positive definite, and the assumption of being divergence free has no clear motivation except that it allows to establish a relationship between $J^{\mu\nu}(x)$ and $W^{\mu\nu}$. At first glance, this may appear as an unsatisfactory feature of the model. Instead, we argue that this difficulty is merely the signal that the quantum current (4.14) is related to an underlying gauge symmetry of a fully fledged quantum field theory of strings. The natural candidate seems to be the generalized gauge symmetry [23]

$$\begin{aligned} \Psi'[C] &= \Psi[C] \exp\left(\frac{i}{\hbar} \oint_C \Lambda_\mu(x) dx^\mu\right) \\ \Psi^{*\prime}[C] &= \Psi^*[C] \exp\left(-\frac{i}{\hbar} \oint_C \Lambda_\mu(x) dx^\mu\right) \\ A'_{\mu\nu}(x) &= A_{\mu\nu}(x) - \frac{1}{g}\partial_{[\mu}\Lambda_{\nu]}\ . \end{aligned} \tag{4.19}$$

Then, the vanishing of $\partial_\mu J^{\mu\nu}(x)$ follows from the transversality of $A_{\mu\nu}$. The symmetry (4.19) allows to choose the *Lorenz gauge* $\partial^\mu A_{\mu\nu} = 0$ to dispose of the unphysical components of the Kalb-Ramond potential and the generalized Maxwell equation requires the two-index current to be divergence free:

$$\partial_\mu H^{\mu\nu\rho}(x) - g^2\overline{\rho}^2(x) A^{\nu\rho}(x) = g J^{\nu\rho}(x)\ , \rightarrow \partial_\mu J^{\mu\nu}(x) = 0\ . \tag{4.20}$$

However, setting this problem aside, we conclude that the self- consistency of our semi-classical stochastic model requires a gauge invariant coupling to a two-form potential $A_{\mu\nu}(x)$, and the interpretation of the loop-functional as a field operator. Remarkably, the classical limit of this loop field theory exists and corresponds to the *statistical* Hamilton–Jacobi–Carathéodorytheory of strings.

V. STOCHASTIC PREGEOMETRY

In this section we wish to take one final step towards the construction of a stochastic approach to the theory of relativistic strings. So far the theory has been formulated in *Minkowski space* and this formulation may well be adequate for particle physics. However, since strings may play an important role in the problem of formation of structure in the universe, or in the very early universe at superunification time, the assumption of a featureless, preexisting spacetime continuum untouched by the dynamical events unfolding in the universe, seems unwarranted. Therefore, our next step is to relax that assumption. The point of view advocated here is that the geometry of spacetime should not be assigned a priori, but should be compatible, as in the case of gravity, with the matter content of the universe. Thus, our specific objective is to establish: i) to what extent the *dynamics* of a string network, interpreted as a stochastic process affects the geometry of spacetime and, conversely ii) how the structure of spacetime affects the distribution of matter in the universe. Interestingly enough, it turns out that the string dynamics as well as the geometry of spacetime are determined by the same averaged least action principle discussed in the previous sections. A result similar to ours was obtained for *point-like particles* by Santamato [6], and our work is a direct extrapolation of that approach to the case of relativistic strings. In a conventional approach to this problem, one would obtain the background geometry of spacetime by integrating away the strings degrees of freedom in the string functional [19]. As an alternative, here we suggest a non minimal coupling between the statistical string ensemble and the curvature of spacetime. The paradigm of this approach is the generalized Nambu-Goto action

$$
\begin{aligned}
I &= -\int_D d^2\xi \left[m^4 + \kappa R(x)\right]^{1/2} \sqrt{-\gamma} - \oint_C dx^\mu S \partial_\mu U \ , \\
&= -\int_D d^2\xi \left[\left(m^4 + \kappa R(x)\right)^{1/2} \sqrt{-\gamma} - \frac{1}{2}\dot{x}^{\mu\nu}\partial_{[\mu} S \partial_{\nu]} U\right] \ , \\
m^2 &\equiv 1/2\pi\alpha' \ , \quad [\kappa] = (length)^{-2} \ .
\end{aligned}
\tag{5.1}
$$

In four dimensions, the *minimal* departure from the assumption of a Riemannian geometry is the weaker assumption that spacetime is a generic manifold with torsion free connections ${\Gamma^\lambda}_{\mu\nu}$ so that the action (5.1) describes the coupling of the string degrees of freedom to both metric and connection through the curvature scalar $R(x) = g^{\mu\nu}(x)R_{\mu\nu}(\Gamma)$. The second term in (5.1) is the now familiar boundary term introduced by Carathéodory's formulation of the least action principle. In this case, the complete Carathéodory figure is defined by the Hamilton-Jacobi equations

$$
\frac{\dot{x}_{\mu\nu}}{\sqrt{-\gamma}} = \frac{\partial_{[\mu} S \partial_{\nu]} U}{\sqrt{m^4 + \kappa R}} \ , \tag{5.2}
$$

$$
\frac{1}{2}\partial_{[\mu} S \partial_{\nu]} U \partial^{[\mu} S \partial^{\nu]} U = -\left(m^4 + \kappa R\right) \ . \tag{5.3}
$$

The solutions of the first equation represent the family of extremal surfaces while the second equation governs the evolution of the wave fronts $S(x) = const.$, $U(x) = const.$ The

passage to a statistical string ensemble follows the same steps described in the previous sections. Since most of the details are the same, we outline the main steps:

1) consider a two–parameter family of world-sheets $x^\mu = x^\mu(\xi^a;\theta^i)$, $i = 1,2$ randomly distributed in parameter space according to a given distribution function $\mu = \mu(\theta^i)$.

2) Averaging over the above parameters, one obtains the action for the statistical string ensemble

$$\langle I\rangle = -\int_D d^2\xi \int d^2\theta\, \mu(\theta^i)\, \left[\left(m^4+\kappa R\right)^{1/2}\sqrt{-\gamma} - \frac{1}{2}\dot{x}^{\mu\nu}\partial_{[\mu}S\partial_{\nu]}U\right] \tag{5.4}$$

3) The action is now defined as an integral over four-dimensional *spacetime* by using the following code of correspondence and transformation rules

$$\begin{aligned}
&x^\mu \leftrightarrow \tau,\sigma,\theta^1,\theta^2\ , \quad \mu(x) = \mu(\theta^i(x))\ ,\\
&d^2\xi\, d^2\theta\, \mu(\theta^i) = d^4x\, Z(x)\,\mu(x)\ , \qquad Z(x) \equiv \frac{\partial(\tau,\sigma,\theta^1,\theta^2)}{\partial(x^0,x^1,x^2,x^3)}\ ,\\
&\rho(x) \equiv Z(x)\,\mu(x)\ , \quad W^{\mu\nu}(x) \equiv \rho(x)\,\dot{x}^{\mu\nu}(x)\ .
\end{aligned} \tag{5.5}$$

The result of this procedure is the following action defined solely in terms of field variables

$$\begin{aligned}
\langle I\rangle &= -\int_D d^4x \left[\left(m^4+\kappa R\right)^{1/2}\sqrt{-\frac{1}{2}W^{\mu\nu}W_{\mu\nu}} - \frac{1}{2}W^{\mu\nu}\partial_{[\mu}S\partial_{\nu]}U\right]\ ,\\
&= -\int_D d^4x \left[\left(m^4+\kappa R\right)^{1/2}\sqrt{-\frac{1}{2}W^{\mu\nu}W_{\mu\nu}} - \frac{1}{2}W^{\mu\nu}\nabla_{[\mu}A_{\nu]}\right]\ .
\end{aligned} \tag{5.6}$$

4) Varying this action with respect to $W^{\mu\nu}$ and A_ν yields the field equations

$$\nabla_\mu W^{\mu\nu} = 0\ , \tag{5.7}$$

$$\left(m^4+\kappa R\right)^{1/2}\frac{W_{\mu\nu}}{\sqrt{-|W|^2}} = \nabla_{[\mu}A_{\nu]}\ . \tag{5.8}$$

Since we have assumed a torsion-free connection, we can replace the covariant derivatives in $F_{\mu\nu}$ above with ordinary derivatives. Then, if the functions $S(x)$ and $U(x)$ satisfy the Hamilton-Jacobi equation (5.3) that is,

$$F_{\mu\nu}F^{\mu\nu} = -2(m^4+\kappa R)\ , \tag{5.9}$$

then $W^{\mu\nu}$ can be identified with the *Gibbs current density* associated with the statistical ensemble

$$W^{\mu\nu} = \frac{1}{m^4}\overline{\rho}(x)\sqrt{-g}\, g^{\mu\rho}g^{\nu\sigma}\, F_{\rho\sigma}\ . \tag{5.10}$$

With this expression for the current density, the action for the string network takes the remarkably simple form

$$\langle I \rangle_{cl.} = -\frac{1}{m^4} \int d^4x \sqrt{-g}\, \overline{\rho}(x) \left[m^4 + \kappa R(x) \right] \tag{5.11}$$

This result is precisely analogous to that obtained by Santamato [6] in the case of a statistical ensemble of relativistic point-like particles. Therefore, we can draw a similar conclusion: variation of the above action with respect to the connection yields

$$\delta_\Gamma \langle I \rangle_{cl.} = 0 \ , \rightarrow \Gamma^\lambda{}_{\mu\nu} = \{{}_\mu{}^\lambda{}_\nu\} + \frac{1}{2} \left(\phi_\mu \delta^\lambda_\nu + \phi_\nu \delta^\lambda_\mu - g_{\mu\nu} g^{\lambda\rho} \phi_\rho \right) \tag{5.12}$$

which represents the anticipated deviation from the Riemannian geometry in the form of a Weyl-Christoffel connection. In all fairness and with hindsight, this result is hardly surprising since it follows whenever a first order, or Palatini method is employed in any model where the scalar curvature is directly coupled to some matter field. The novelty of our result is that the $\overline{\rho}(x)$ field, rather than being a generic field variable in a scalar–tensor theory of gravity, represents the macroscopic string energy density. With this specific interpretation of the $\overline{\rho}(x)$ field in mind, there are two aspects of the result (5.12) which are worth emphasizing. First, the *Weyl field* $\phi_\mu(x)$ is longitudinal, or pure gauge, since

$$\phi_\mu(x) = \overline{\rho}^{\,-1} \partial_\mu \overline{\rho} = \partial_\mu \ln(\overline{\rho}/M^4) \ , \tag{5.13}$$

where M represents an arbitrary constant. This is not an assumption but a dynamical consequence of the theory. Therefore the length of a vector remains constant after parallel displacement along a closed path

$$L^2 = L_0^2 \exp \left(\oint_\gamma dx^\mu \phi_\mu(x) \right) = L_0^2 \ , \tag{5.14}$$

thereby removing, at least in this theory, one of the main criticisms about the physical relevance of Weyl's geometry. The second and most important point is that this new-found geometry does not extend over the entire spacetime; rather, it is confined to those regions where the string density fluctuates or changes appreciably. In contrast, wherever $\overline{\rho}(x) \simeq const. \equiv \overline{\rho}_0 \rightarrow \phi_\mu \simeq 0$ and our classical action reduces to Einstein's action with a cosmological constant

$$\langle I \rangle_{cl.} = -\frac{1}{m^4} \int d^4x \sqrt{-g}\, \overline{\rho}_0 \left[m^4 + \kappa R(g) \right] \ , \tag{5.15}$$

if we identify $R(g)$ with Ricci's *Riemannian* scalar curvature and

$$\frac{\kappa \overline{\rho}_0}{m^4} \equiv \frac{1}{16\pi G_N} \ , \qquad \overline{\rho}_0 = -\frac{\Lambda}{8\pi G_N} \ . \tag{5.16}$$

We would like to conclude this section, and the paper, with few speculative remarks on the possible consequences of the above results on the longstanding problem of formation of structure in the universe. According to our discussion, spacetime seems to have a multiphase, or *cellular* structure such that regions in a Riemannian geometric phase in which the string density is roughly constant, are connected by Weyl regions, or *domain walls* over which the string density changes abruptly. The Weyl geometry appears to be the geometry of the

domain walls between Riemannian cells of spacetime. One possible picture that this structure brings to mind is the observed*soap–bubble–like* pattern of cosmic voids with vanishing energy density, separated by domain walls on which filamentary, or string-like superclusters of galaxies seem to be concentrated. Admittedly, this is a purely qualitative picture. However, it emerges from a completely analytical approach to the dynamics of a string network. Of course, an in–depth analysis is needed, possibly using computer simulations, in order to corroborate its validity.

Another possible scenario that the cellular structure of spacetime brings to mind, is that of *chaotic inflation*. Following Linde [24], in this case we envisage the whole Universe as a cluster of microuniverses, some of them inflationary depending on the value and sign of the string energy density as given by equation (5.16). A typical initial size of a randomly chosen spacetime cell should be of the order of Planck's length with an energy density of the order of Planck's density. The classical evolution of any such cell has been analyzed in detail in an earlier paper [25]. The present paper goes one step further, at least at a conceptual level: the controlling factor here is the stochastic field $\overline{\rho}(x)$ which appears in the expression of Weyl's field and represents the string density in spacetime. We cannot overemphasize the fact that this stochastic field originates at the quantum level and represents the classical counterpart of the probability density $|\Psi[C]|^2$ in loop space. Thus, the *dynamically induced* Weyl-Riemann geometry of spacetime that we have uncovered, should be regarded as the result of an averaging process over the string quantum fluctuations and, ultimately, it may originate from the stochastic nature of spacetime itself. In this sense, the present work may well provide a quantum basis for stochastic inflation. Finally, to the extent that the classical limit of the Wheeler-DeWitt string wave equation exists and that this limit represents a *statistical* theory, we claim that we have developed also a *stochastic* quantum string theory.

REFERENCES

[1] E.J.Heller and S.Tomsovic, Phys. Today, 38, (July 1993) .
[2] Y.Nambu, in *Symmetries and Quark Models*, edited by R.Chand. (Gordon and Breach,NY,1970 .
[3] H.B.Nielsen and P.Olesen, Nucl. Phys. **B57**, (1973), 367 .
[4] Daren Austin,E.J.Copeland and T.W.B.Kibble, Phys.Rev. **D48**, 5594, (1993) .
[5] In this connection see also, F.Embacher, Nucl. Phys. **B387**, 129, (1992) .
[6] E.Santamato, Phys. Rev. **D29**, 216, (1984) ; Phys. Rev. **D32**, 2615, (1985) .
[7] T.Eguchi, Phys. Lett. **59B**, 73, (1975) .
[8] Y.Nambu, Phys. Rep. **23C**, 250, (1976) .
[9] Y.Nambu, Phys. Lett. **80B**, 372, (1979) .
[10] Y.Nambu, Phys. Lett. **92B**, 327, (1980) .
[11] Y.Nambu, Phys. Lett. **102B**, 149, (1981) .
[12] Y.Hosotani, Phys. Rev. Lett. **55**, 1719, (1985) .
[13] A.Aurilia, S.Smailagic and E.Spallucci, Phys. Rev. **D47**, 2563, (1993) .
[14] A.Aurilia and E.Spallucci Class. Quantum. Grav. **10**, 1217, (1993) .
[15] J.Stachel, Phys. Rev. **D21**, 2171, (1980).
[16] A.Schild, Phys. Rev. **D16**, 1722, (1977) .
[17] H.A.Kastrup and M.Rinke, Phys. Lett. **105B**, 191, (1981) .
[18] H.A.Kastrup, Phys. Rep. **101**, 1, (1983)
[19] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. **B261**, 1, (1985) .
[20] A.A. Migdal, Phys. Rep. **102**, 199, (1983) .
[21] C. Rovelli and L.Smolin, Nucl. Phys. **B331** (1990), 80.
[22] C.Marshall and P.Ramond, Nucl. Phys. **B85**, 375, (1975)
[23] M.Kalb and P.Ramond, Phys. Rev. **D9**, 2273, (1973) .
[24] A.Linde, Phys. Today **40**, 61, (1987) .
[25] A.Aurilia, M.Palmer and E.Spallucci, Phys. Rev. **D40**, 2511, (1989) .